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THE ACTION-ANGLE VARIABLES IN THE EULER-POINSOT PROBLEM

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Use of the action-angle variables (see e. g. [1]) leads, in a number of cases, to considerable simplification when the perturbation method is applied to study the dynamics of perturbed motion, especially when computing the higher order approximations. Below we obtain such variables for the problem of a solid rotating freely about a fixed point (the Euler-Poinsot case).

Free motion of a solid with a fixed point can be described by a system of canonical equations whose Hamiltonian is [2]

$$H = \frac{G^2 - G_\zeta^2}{2}\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) + \frac{G_\zeta^2}{2C} \tag{1}$$

Here A, B, C are the principal moments of inertia of the body relative to the fixed point, G is the kinetic moment, G_ζ is its projection on the axis corresponding to the moment of inertia C of the associated coordinate system, and ψ, ϑ, φ are the Euler angles (of precession, nutation and self-rotation) defining the position of the body in the fixed coordinate system of which one axis is collinear with the kinetic moment vector. Position of this vector in the initial absolute coordinate system can be defined by the following two quantities: L which is the projection of the kinetic moment on one axis of the initial coordinate system, and the angle h. The quantities $G, G_\zeta, L, \psi, \varphi, h$ form a complete set of canonical variables for the present problem.

Change to the action-angle variables is effected by means of a canonical transformation which transforms the Hamiltonian H into a function of impulses only, and is independent of the angles.

In our problem we can use the triad G, L, I of impulses as the action variables. Here I is the projection of the kinetic moment on an axis of the associated coordinate system, averaged over the characteristic rotation

$$I = \frac{1}{2\pi}\oint G_\zeta \, d\varphi \tag{2}$$

Here the integration is performed over the complete cycle of variation of G_ζ relative to φ, in accordance with Eq. (1). If we choose as the $O\zeta$ axis of the associated coordinate system that axis of the ellipsoid of inertia on which the projection of the kinetic moment vector is always positive, we can express I in terms of the initial variables as follows:

$$I = \frac{2G}{\pi\varkappa}\left(\frac{1+\varkappa^2}{\lambda^2+\varkappa^2}\right)^{1/2}\left[(\lambda^2+\varkappa^2)\,\Pi\left(\frac{\pi}{2}, \varkappa^2, \lambda\right) - \lambda^2 K(\lambda)\right] \tag{3}$$

where $\varkappa$ and λ are positive parameters given by

$$\varkappa^2 = \frac{C(B-A)}{A(C-B)}, \qquad \lambda^2 = \varkappa^2\frac{2CH-G^2}{G^2-2AH} \tag{4}$$

Relation (3) defines λ as an implicit function of I/G. This enables us to obtain the required angular variables using the following generating function:

$$S = Lh + G\psi + \int G_\zeta(G, I, \varphi)\, d\varphi$$

On performing the necessary calculations we find that the angles conjugated with the impulses L, G, I are, respectively,

$$h = \frac{\partial S}{\partial L}$$

$$v = \frac{\partial S}{\partial G} = \psi + \frac{\sqrt{(1+\varkappa^2)(\lambda^2+\varkappa^2)}}{\varkappa}\left[\Pi(\xi, \varkappa^2, \lambda] - \Pi\left(\frac{\pi}{2}, \varkappa^2, \lambda\right)\frac{F(\xi, \lambda)}{K(\lambda)}\right]$$

$$f = \frac{\partial S}{\partial I} = \frac{\pi}{2}\frac{F(\xi, \lambda)}{K(\lambda)}$$

The auxilliary variable ξ is related to φ as follows:

$$\operatorname{ctg}\varphi = -\sqrt{1+\varkappa^2}\ \operatorname{tg}\xi \quad (\ \xi = 0 \ \text{ for } \ \varphi = \pi/2\ )$$

The Hamiltonian of the unperturbed motion assumes the following form in the action-angle variables L, G, I, h, v, f:

$$H = \frac{G^2}{2A}\left(1 - \frac{C-A}{C}\frac{\varkappa^2}{\varkappa^2+\lambda^2}\right)$$

where λ (in accordance with (3)), should be regarded as a function of I/G.

Equations of the unperturbed motion

$$L^\cdot = 0,\ G^\cdot = 0,\ I^\cdot = 0,\ h^\cdot = 0$$

$$v^\cdot = \frac{\partial H}{\partial G} = Q_1(I, G), \qquad f^\cdot = \frac{\partial H}{\partial I} = Q_2(I, G)$$

are easily integrable.

In order to be able to study the perturbed motion using the variables just obtained we must express the corresponding Hamiltonians in terms of these variables. Since the perturbation part of the Hamiltonian is usually a function of body position which is defined by the direction cosines if the associated coordinate system is the absolute one, it is sufficient to express the direction cosines of the body in terms of the new variables.

We find that the matrix of the direction cosines can be written in the form

$$S = S_1 S_2 S_3$$

where

$$S_1 = \begin{Vmatrix} \cos h & -\sin h \cos\delta & \sin h \sin\delta \\ \sin h & \cos h \cos\delta & -\cos h \sin\delta \\ 0 & \sin\delta & \cos\delta \end{Vmatrix} \qquad \left(\cos\delta = \frac{L}{G} > 0\right)$$

$$S_2 = \begin{Vmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

and the elements of the matrix S_3 depend on a single angular variable f, and can therefore be expanded into Fourier series in this variable. The formulas obtained are analogous to those obtained by Jacobi while deriving the explicit relation between the direction cosines and time [3]. The following expansions are valid for the elements s_{ij} of the matrix S_3:

$$s_{11} = -\frac{2\pi}{K}\frac{1}{\sqrt{\varkappa^2+\lambda^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}(1-q^{2n+1})\operatorname{ch}\sigma}{1-2q^{2n+1}\operatorname{ch}2\sigma+q^{4n+2}}\sin(2n+1)f$$

$$s_{12} = -\frac{2\pi}{K}\sqrt{\frac{1+\varkappa^2}{\lambda^2+\varkappa^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}(1+q^{2n+1})\operatorname{ch}\sigma}{1+2q^{2n+1}\operatorname{ch}2\sigma+q^{4n+2}}\cos(2n+1)f$$

$$s_{13} = -\frac{2\pi}{K}\frac{\varkappa}{\sqrt{\varkappa^2+\lambda^2}}\sum_{n=1}^{\infty}\frac{q^{n}(1-q^{2n})\operatorname{ch}\sigma}{1-2q^{2n}\operatorname{ch}2\sigma+q^{4n}}\sin 2nf$$

$$s_{21} = \frac{2\pi}{K}\frac{1}{\sqrt{\varkappa^2+\lambda^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}(1+q^{2n+1})\operatorname{sh}\sigma}{1-2q^{2n+1}\operatorname{ch}2\sigma+q^{4n+2}}\cos(2n+1)f$$

$$s_{22} = -\frac{2\pi}{K}\sqrt{\frac{1+\varkappa^2}{\lambda^2+\varkappa^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}(1-q^{2n+1})\operatorname{sh}\sigma}{1+2q^{2n+1}\operatorname{ch}2\sigma+q^{4n+2}}\sin(2n+1)f$$

$$s_{23} = \frac{2\pi}{K}\frac{\varkappa}{\sqrt{\varkappa^2+\lambda^2}}\left\{-\frac{1}{4\operatorname{sh}\sigma}+\sum_{n=1}^{\infty}\frac{q^{n}(1+q^{2n})\operatorname{sh}\sigma}{1-2q^{2n}\operatorname{ch}2\sigma+q^{4n}}\cos 2nf\right\}$$

$$s_{31} = \frac{2\pi}{K}\frac{1}{\sqrt{\varkappa^2+\lambda^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos(2n+1)f$$

$$s_{32} = -\frac{2\pi}{K}\sqrt{\frac{1+\varkappa^2}{\varkappa^2+\lambda^2}}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin(2n+1)f$$

$$s_{33} = \frac{2\pi}{K}\frac{\varkappa}{\sqrt{\varkappa^2+\lambda^2}}\left(\frac{1}{4}+\sum_{n=1}^{\infty}\frac{q^{n}}{1+q^{2n}}\cos 2nf\right)$$

Here $K \equiv K(\lambda)$ is the complete elliptic integral of the first kind with modulus λ, $q = \exp(-\pi K'/K)$ and the parameter σ is given by

$$\sigma = \frac{\pi}{2K}F\left(\operatorname{arc\,tg}\frac{\varkappa}{\lambda},\ \sqrt{1-\lambda^2}\right)$$

The computing procedures and results are given in more detail in [4].

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OPTIMAL STABILIZATION OF ROTATION OF A GYROSTAT IN THE NEWTONIAN FORCE FIELD

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We solve the problem of optimal (in a certain defined sense) stabilization of rotation of a gyrostat (a rigid body with three flywheels) whose center of mass moves along a circular orbit in the central Newtonian force field.

In [1, 2] an analogous problem of stabilization of rotation of a rigid body in inertial motion was solved. Problems of stability of positions of relative equilibrium of stationary motions of rigid bodies and gyrostats in the Newtonian force field were studied in detail in [3–6]. We know that the motions of a rigid body mentioned above can be stabilized by passive damping [7, 8].

1. Initial equations of motion. Statement of the problem. Using the notation of [1] we shall consider a symmetrical gyrostat, i. e. a rigid body with three flywheels ($C_1 = C_2 = C$, $I_1 = I_2 = I$) moving in the central Newtonian force field (O_1 is the center of attraction and O is the center of mass of the gyrostat). Equations of motion of the gyrostat [4, 5] admit the following particular solution of the type of regular precession: the center of mass O moves in the $X_1O_1X_2$ plane along a circular orbit of radius R_0 with constant angular velocity $\Phi^{\cdot} = \omega_1$. The gyrostat rotates uniformly with relative angular velocity $\varphi^{\cdot} = \omega$ about the axis of symmetry Ox_3 normal to the orbital plane. Two flywheels whose axes lie in the plane x_1Ox_2 are at rest, and the third flywheel whose axis of rotation is Ox_3 is either at rest or in uniform motion relative to the body. Figure 1 and Table 1 depict the following coordinate systems: $O_1X_1X_2X_3$ is inertial; $Ox_1x_2x_3$ is rigidly connected with the gyrostat and its axes coincide with the axes

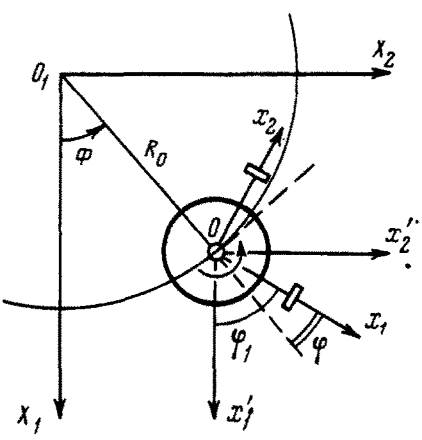

Fig. 1

Table 1

	x_1'	x_2'	x_3'
X_1	β_{11}	β_{12}	β_{13}
X_2	β_{21}	β_{22}	β_{23}
X_3	β_{31}	β_{32}	β_{33}